Name: Key

Review for Test on 5.1, 5.2, 5.4, 5.5, 6.2, and 6.3

$$\frac{d}{dx}\left[a^u\right]=(\ln a)a^u\frac{du}{dx} \qquad \frac{d}{dx}\left[\log_a u\right]=\frac{1}{(\ln a)u}\frac{du}{dx}$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

$$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$$

Find the derivative.

1. $y = \left(e^x + e^{3x}\right)^4$ $\qquad$ $y' = 4\left(e^x+3e^{3x}\right)\left(e^x+e^{3x}\right)^3$

$y' = 4\left(e^x+e^{3x}\right)^3\left(e^x+3e^{3x}\right)$

2. $y = \ln\sqrt{x^2-9}$ $\qquad$ $y' = \dfrac{x}{x^2-9}$

$y = \frac{1}{2}\ln\left(x^2-9\right)$

$y' = \frac{1}{2}\cdot\frac{1}{x^2-9}\cdot 2x$

3. $y = 8^{x-2}$ $\qquad$ $y' = (\ln 8)8^{x-2}$

$y' = \ln 8 \cdot 8^{x-2}\cdot 1$

4. $y = \ln\left(\dfrac{1-x}{1+x}\right) = \ln(1-x) - \ln(1+x)$ $\qquad$ $y' = \dfrac{2}{x^2-1}$ $\qquad$ same as $y = \dfrac{-2}{1-x^2}$

$y' = \frac{1}{1-x}\cdot(-1) - \frac{1}{1+x}(1) = \frac{-1}{1-x} - \frac{1}{1+x} = \frac{-(1+x)-(1-x)}{(1-x)(1+x)} = \frac{+2}{(x-1)(x+1)}$

5. $y = e^{\sin x^3}$ $\qquad$ $y' = 3x^2\cos\left(x^3\right)e^{\sin\left(x^3\right)}$

$y' = e^{\sin\left(x^3\right)}\cdot\cos\left(x^3\right)\cdot 3x^2$

6. $y = x^{\sqrt{x}}$

$\ln y = \ln x^{\sqrt{x}}$

$\ln y = \sqrt{x} \ln x$

$\frac{1}{y} \cdot y' = \sqrt{x}\left(\frac{1}{x}\right) + \frac{1}{2} x^{-1/2} \ln x$

$\frac{y'}{y} = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}$

$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right)$

$y' = x^{\sqrt{x}}\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right)$

7. $y = \log\left[3x(x+4)^6\right]$

$y = \log_{10}(3x) + 6\log_{10}(x+4)$

$y' = \frac{1}{\ln 10} \cdot \frac{1}{3x} \cdot 3 + 6 \cdot \frac{1}{\ln 10} \cdot \frac{1}{x+4} \cdot 1 = \frac{1}{x \ln 10} + \frac{6}{(\ln 10)(x+4)}$

$y' = \frac{1}{x\ln 10} + \frac{6}{\ln 10(x+4)} = \frac{7x+4}{(\ln 10) x (x+4)}$

8. $y = e^{-x}\cos(2x)$

$y' = e^{-x}(-\sin(2x)(2)) + (-e^{-x})\cos(2x)$

$= -2e^{-x}\sin(2x) - e^{-x}\cos(2x)$

$= -e^{-x}(2\sin(2x) + \cos(2x))$

$y' = -e^{-x}(2\sin 2x + \cos 2x)$

9. $y = \cot(\ln x)$

$y' = -\csc^2(\ln x) \cdot \frac{1}{x} \cdot 1$

$y' = \frac{-\csc^2(\ln x)}{x}$

10. Find $\frac{dy}{dx}$ using logarithmic differentiation if $y = \frac{(x+2)\sqrt{1-x^2}}{4x^3}$. <take log of both sides>

$\ln y = \ln(x+2) + \frac{1}{2}\ln(1-x^2) - \ln 4 - 3\ln x$

$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x+2} + \frac{1}{2} \cdot \frac{1}{1-x^2} \cdot -2x - 0 - \frac{3}{x}$

$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x+2} - \frac{x}{1-x^2} - \frac{3}{x} =$

$\frac{dy}{dx} = y\left[\frac{1}{x+2} - \frac{x}{1-x^2} - \frac{3}{x}\right]$

$\frac{dy}{dx} = \frac{x+2\sqrt{1-x^2}}{4x^3}\left(\frac{1}{x+2} - \frac{x}{1-x^2} - \frac{3}{x}\right)$

11. Find $\frac{dy}{dx}$ given $\ln(xy) = x + y$.

$\frac{1}{xy} \cdot [x \cdot y' + 1 \cdot y] = 1 + y'$

$\frac{xy'}{xy} + \frac{y}{xy} = 1 + y'$

$\frac{y'}{y} + \frac{1}{x} = 1 + y'$

$y'\left(\frac{1}{y} - 1\right) = 1 - \frac{1}{x}$

$y' = \frac{1 - \frac{1}{x}}{\frac{1}{y} - 1} = \frac{\frac{x-1}{x}}{\frac{1-y}{y}}$

$y' = \frac{(x-1)y}{x(1-y)}$

$y' = \frac{-y(x-1)}{x(y-1)}$

12. Find the value of the integral $\int_0^1 \frac{e^x}{e^x+1}dx$. $\ln\left(\frac{1+e}{2}\right)$

$u = e^x + 1$
$du = e^x dx$

$\int_2^{1+e} \frac{1}{u}\,du = \ln|u| \Big|_2^{1+e}$

$\ln(1+e) - \ln 2$

13. Find the value of the integral $\int_0^1 e^{-2x}\,dx$. $\frac{e^2-1}{2e^2}$ or $\frac{-1}{2e} + \frac{1}{2}$

$u = -2x$
$du = -2dx$

$-\frac{1}{2}\int_0^{-2} e^u du = -\frac{1}{2} e^u \Big|_0^{-2} = -\frac{1}{2}\left[e^{-2} - e^0\right] = -\frac{1}{2}\left[\frac{1}{e^2} - 1\right]$

$= \frac{-(1-e^2)}{2e^2}$ or $\frac{-1}{2e} + \frac{1}{2}$

14. Evaluate $\int 4^{-x}\,dx$. $\Rightarrow -\int 4^u du$ $-\frac{1}{\ln 4}\cdot 4^{-x} + C$

$u = -x$
$du = -dx$

$= -\frac{1}{\ln 4}\cdot 4^u + C$

15. Solve the differential equation $\frac{dy}{dx} = x\cos(x^2)$. $y = \frac{1}{2}\sin(x^2) + C$

$\int dy = \int x\cos(x^2)\,dx$

$u = x^2$
$du = 2x\,dx$

$\int dy = \frac{1}{2}\int \cos u\,du$

$y = \frac{1}{2}\sin u + C$

16. Solve the differential equation $y' = x(1 + y)$. $y = Ce^{\frac{1}{2}x^2} - 1$

$\int \frac{dy}{1+y} = \int x\,dx$ $u = 1+y$ $du = dy$

$1 + y = \pm e^{C_1}\cdot e^{\frac{1}{2}x^2}$ let $C = \pm e^{C_1}$

$\ln|1+y| = \frac{1}{2}x^2 + C_1$

$|1+y| = e^{\frac{1}{2}x^2 + C_1}$

$y = Ce^{\frac{1}{2}x^2} - 1$

17. Find the function y = f(t) passing through the point (0, 10) with the given first derivative or slope: $\frac{dy}{dt} = -\frac{1}{2}y$.

$y = 10e^{-t/2}$

since $y' = ky$, $y = Ce^{kt}$

$k = -\frac{1}{2}$

$y = Ce^{-\frac{1}{2}t}$

$10 = Ce^{-\frac{1}{2}(0)}$

$C = 10$

Separation of Variables

18. Solve the differential equation xyy ′ - lnx = 0 for the initial condition y(1) = 2.

$xy\frac{dy}{dx} = \ln x$

$\int y\,dy = \int \frac{\ln x}{x}\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + C_1$

$y^2 = (\ln x)^2 + C$

$\int \frac{\ln x}{x}\,dx = \int u\,du$

$u = \ln x$

$du = \frac{1}{x}dx$

$\frac{1}{2}u^2 + C$

$\frac{1}{2}(\ln x)^2 + C$

$4 + C = 2C_1$

$y^2 = (\ln x)^2 + 4$

$y^2 = (\ln x)^2 + C$

$(2)^2 = (\ln 1)^2 + C$

$C = 4$

19. Find the particular solution that satisfies the initial conditions: $f''(x) = \sin x + e^{2x}$, $f'(0) = \frac{1}{2}$, $f(0) = \frac{1}{4}$.

$f(x) = -\sin x + \frac{1}{4}e^{2x} + x$

$f'(x) = \int \sin x + e^{2x}\,dx$

$= \int \sin x\,dx + \int e^{2x}\,dx$

$u = 2x \quad du = 2dx$

$f'(x) = -\cos x + \frac{1}{2}e^{2x} + C_1$

$\frac{1}{2} = -\cos(0) + \frac{1}{2}e^{2(0)} + C_1$

$= -1 + \frac{1}{2} + C_1 \quad \therefore C = 1$

$f(x) = \int(-\cos x + \frac{1}{2}e^{2x} + 1)\,dx$

$f(x) = -\sin x + \frac{1}{4}e^{2x} + x + C_2$

$\frac{1}{4} = -\sin(0) + \frac{1}{4}e^{2(0)} + 0 + C_2$

$\frac{1}{4} = \frac{1}{4} + C_2 \quad \therefore C_2 = 0$

20. A population of bacteria is changing at the rate of $\frac{dp}{dt} = \frac{2000}{1+0.2t}$, where t is the time in days. The initial population is 1000.

a. Write an equation that gives the population at any time t. $P = 10{,}000\ln(1+.2t) + 1{,}000$

$P = \int \frac{2000}{1+.2t}\,dt$

$u = 1 + .2t$

$du = .2dt$

$= 2000(5)\int \frac{1}{u}\,du$

$= \frac{1}{5}dt$

$P = 10{,}000\ln|1+.2t| + C$

$(0, 1000)$

$C = 1000$

b. Find the population after 10 days.

≈ 11986 bacteria

$P(10) = 10{,}000\ln(1 + .2(10)) + 1{,}000$

$10{,}000\ln 3 + 1{,}000$

≈ 11986.1229

21. A certain type of bacteria increases so that its rate is directly proportional to the number present. If there are initially 500 bacteria present and 1000 bacteria two hours later, how many will there be five hours from the initial time given?

$y = Ce^{kt}$

$C = 500$

$(0, 500)$

$(2, 1000)$

$1000 = 500e^{k(2)}$

$2 = e^{2k}$

$\ln 2 = \ln e^{2k}$

$\ln 2 = 2k$

$k = \frac{\ln 2}{2}$

$y = 500e^{\frac{t\ln 2}{2}}$

≈ 2828 bacteria

when $t = 5$

$y = 500e^{\frac{5\ln 2}{2}} \approx 2828.43$